

# APPLIED CHIRAL PERTURBATION THEORY

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## ABSTRACT

I consider some selected topics in chiral perturbation theory (CHPT) as probed at colliders such as DAΦNE. Emphasis is put on processes involving pions in the isospin zero S-wave which require multi-loop calculations. These include the scalar form factor of the pion, two-photon fusion into pion pairs and  $K_{\ell 4}$ -decays. The physics of the chiral anomaly is briefly touched upon

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# 1. INTRODUCTION

This talk will be concerned with certain aspects of the standard model in the long-distance regime. I will argue that there exists a rigorous calculational scheme and that plenty of interesting and *fundamental* problems await a solution. Particular emphasis is put on reactions to be measured at the  $\phi$ -factory DAΦNE or at other places where intense kaon fluxes are available (like e.g. Brookhaven). There are many accurate predictions of chiral perturbation theory which await detailed tests. The bottom line is that these low energy reactions will tell us about our understanding of the mechanism of spontaneous chiral symmetry breaking in QCD and also lead to a rich phenomenology.

Our starting point is the observation that in the three flavor sector, the QCD Hamiltonian can be written as

$$\begin{aligned} H_{\text{QCD}} &= H_{\text{QCD}}^0 + H_{\text{QCD}}^I \\ H_{\text{QCD}}^I &= \int d^3x \{m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s\} \end{aligned} \tag{1}$$

with  $H_{\text{QCD}}^0$  symmetric under chiral  $\text{SU}(3)_L \times \text{SU}(3)_R$ . On a typical hadronic scale, say  $M_\rho = 770$  MeV, the current quark masses  $m_q = m_u, m_d, m_s$  can be considered as perturbations. The chiral symmetry of the Hamiltonian is spontaneously broken down to its vectorial subgroup  $\text{SU}(3)_V$  with the occurrence of eight (almost) massless pseudoscalar mesons, the Goldstone bosons ( $\varphi = \pi^+, \pi^0, \pi^-, K^+, K^-, K^0, \bar{K}^0, \eta$ )

$$M_\varphi^2 = m_q B + \mathcal{O}(m_q^2) \tag{2}$$

with  $B = -\langle 0|\bar{q}q|0\rangle/F_\pi^2$  and  $F_\pi \simeq 93$  MeV the pion decay constant. Clearly, from eq.(2) one gets immediately some information about the ratios of the light quark masses,  $m_u/m_d = 0.66$ ,  $m_d/m_s = 1/20.1$  and  $2m_s/(m_u + m_d) = 24.1$  (modulo higher order and electromagnetic corrections, see also section 2). In the confinement (long-distance) regime, the properties of the standard model related to this symmetry can be unambiguously worked out in terms of an effective Lagrangian,

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{eff}}[U, \partial_\mu U, \dots, \mathcal{M}] \tag{3}$$

with  $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$  the quark mass matrix and the Goldstone bosons are collected in the matrix-valued field  $U(x) = \exp\{i \sum_{a=1}^8 \varphi_a(x) \lambda^a / F_\pi\}$ . Of course, there is an infinity of possibilities of representing the non-linearly realized chiral symmetry. While the QCD Lagrangian is formulated in terms of quark and gluon fields and the

rapid rise of the strong coupling constant  $a_S(Q^2)$  with decreasing  $Q^2$  forbids a systematic perturbative expansion, matters are different for the effective field theory (EFT) based on the effective Lagrangian (3). It can be written as a string of terms with increasing dimension,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(4)} + \mathcal{L}_{\text{eff}}^{(6)} + \dots \quad (4)$$

if one counts the quark masses as energy squared. To lowest order, the effective Lagrangian contains two parameters,  $F_\pi$  and  $B$ . It is worth to stress that  $B$  never appears alone but only in combination with the quark mass matrix, alas the pseudoscalar meson masses. Consequently, any matrix-element  $\langle ME \rangle$  for the interactions between the pseudoscalars can be written as

$$\langle ME \rangle = c_0 \left(\frac{E}{\Lambda}\right)^2 + \left[ \sum_{i=1}^n (c_{1i}) + (\text{non-local}) \right] \left(\frac{E}{\Lambda}\right)^4 + \mathcal{O}\left(\frac{E}{\Lambda}\right)^6 \quad (5)$$

This is obviously an energy expansion or, more precisely, a simultaneous expansion in small external momenta *and* quark masses. The first term on the r.h.s. of (5) leads to nothing but the well-known current algebra results, the pertinent coefficient  $c_0$  can be entirely expressed in terms of  $F_\pi$ , the Goldstone masses and some numerical constants. As one of the most famous examples I quote Weinberg's result for the S-wave, isospin zero scattering length [1],

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} \quad (6)$$

which is such an interesting observable because it vanishes in the chiral limit  $M_\pi \rightarrow 0$ . At next-to-leading order, life is somewhat more complicated. As first shown by Weinberg [2] and discussed in detail by Gasser and Leutwyler [3], one has to account for meson loops which are naturally generated by the interactions. These lead to what I called "non-local" in (5). In fact, it can be shown straightforwardly that any N-loop contribution is suppressed with respect to the leading order result by  $(E/\Lambda)^{2N}$ . At  $\mathcal{O}(E^4)$ , the loop contributions do not introduce any new parameters. However, one also has to account for the contact terms of dimension four which are accompanied by a priori unknown coupling constants (the  $c_{1i}$  in (5)). These so-called low-energy constants serve to renormalize the infinities related to the pion loops. Their finite pieces are then fixed from some experimental input. In the case of flavor SU(2), one has  $n = 7$ . Two of these constants are related to interactions between the pseudoscalars, three to quark mass insertions and the remaining two have to be determined from current matrix elements. The inclusion of gauge boson couplings to the Goldstone bosons is most simply and economically done

in the framework of external background sources. Notice also that at order  $E^4$  the chiral anomaly can be unambiguously included in the EFT. At order  $E^6$ , one has to consider loop diagrams with insertions from  $\mathcal{L}_{\text{eff}}^{(2)}$  and  $\mathcal{L}_{\text{eff}}^{(4)}$  as well as contact terms from  $\mathcal{L}_{\text{eff}}^{(6)}$  which introduces new couplings. Once the low-energy constants are fixed, the aspects of the dynamics of the standard model related to the chiral symmetry can be worked out *systematically* and *unambiguously*. Clearly, the EFT can only be applied below a typical scale  $\Lambda \simeq M_\rho$  and higher loop calculations become more and more cumbersome (but can't always be avoided as will be discussed below). This is the basic framework of CHPT in a nutshell. For more details, I refer to refs.[2,3], my review [4] and the extensive list of references given therein. It is worth pointing out that Leutwyler has recently given a more sound foundation of the effective Lagrangian approach by relating it directly to the pertinent Ward-Identities [5].

## 2. MESON-MESON SCATTERING AND THE MODE OF QUARK CONDENSATION

Pion-pion and pion-kaon scattering are the purest reactions between the pseudoscalar Goldstone bosons. The Goldstone theorem mandates that as the energy goes to zero, the interaction between the pseudoscalars vanishes. Consequently,  $\pi\pi$  and  $\pi K$  scattering are the optimal testing grounds for CHPT.

Let me first consider the chiral expansion of the isospin zero S-wave in  $\pi\pi$  scattering. In the standard formulation of CHPT, Gasser and Leutwyler have derived a low-energy theorem generalizing Weinberg's result (6) [6],

$$a_0^0 = 0.20 \pm 0.01 \quad (7)$$

which is compatible with the data,  $a_0^0 = 0.23 \pm 0.08$  [7]. The theoretical value (7) rests on the assumption that  $B$  is large, i.e. of the order of 1 GeV (from current values of the scalar quark condensates). However, if  $B$  happens to be small, say of the order of  $F_\pi$ , one has to generalize the CHPT framework as proposed by Stern et al.[8]. In that case, the quark mass expansion of the Goldstone bosons takes the form

$$M_\varphi^2 = m_q B + m_q^2 A + \mathcal{O}(m_q^3) \quad (8)$$

with the second term of comparable size to the first one. In ref.[9], this framework is discussed in more detail and a novel representation of the  $\pi\pi$  amplitude which is exact including order  $E^6$  and allows to represent the *whole*  $\pi\pi$  scattering amplitude in terms of the S- and P-waves and six subtraction constants is given. The presently

available data are not sufficiently accurate to disentangle these two possibilities. More light might be shed on this when the  $\phi$ -factory DAΦNE will be in operation (via precise measurements of  $K_{\ell 4}$ -decays, see section 5) or if the proposed experiment to measure the lifetime of pionic molecules [10] will be done. It should also be pointed out that recent lattice QCD results seem to be at variance with the expansion (8), but this can only be considered as indicative [11]. Also, the experimentally well-fulfilled GMO relation for the pseudoscalar meson masses arises naturally in the conventional CHPT framework but requires parameter fine-tuning in case of a small value of  $B \sim 100$  MeV. Novel high precision experiments at *low energies* are called for. This is an important question concerning our understanding of the standard model and it definitively should deserve more attention. For more details, I refer to sections 4.1 and 4.2 of ref.[4] as well as ref.[9].

In fig.1, I show the phase-shift  $\delta_0^0$  from threshold (280 MeV) to approximately 600 MeV [12]. One notices the rapid rise of the phase shift, and at 600 MeV it is already as large as 55 degrees and passes through 90 degrees at about 850 MeV. At energies below 600 MeV, the other partial waves do not exceed 15 degrees (in magnitude). This behaviour of  $\delta_0^0$  is attributed to the so-called strong pionic final state interactions which I will discuss in section 3.

As indicated in fig.1, beyond 450 MeV the one loop corrections are half as big as the tree phase. Nevertheless, one can make a rather precise statement about the phase of the CP-violation parameter  $\epsilon'$ [12],

$$\Phi(\epsilon') = \frac{\pi}{2} - (\delta_0^0 - \delta_0^2) \Big|_{s=M_{K^0}^2} = (45 \pm 6)^\circ \quad (9)$$

This is due to the fact that the corrections to  $\delta_0^2$  are of the same sign as the ones to  $\delta_0^0$  and thus cancel. At tree level,  $\Phi(\epsilon') = 37^\circ$ . The accuracy of the theoretical prediction is as good as the recent empirical one,  $\Phi(\epsilon')_{\text{exp}} = (43 \pm 8)^\circ$  [7]. Notice that it is much more difficult to get a precise number on  $\Phi(\epsilon')$  from  $K \rightarrow 2\pi$  decays because of the variety of isospin breaking effects one has to account for (this theme is touched upon in ref.[12]).

I briefly turn to the case of  $\pi K$  scattering. Here, the empirical situation is even worse, which is very unfortunate. In the framework of conventional CHPT, the threshold behaviour of the low partial waves can be unambiguously predicted [13] since all low-energy constants in SU(3) are fixed. Furthermore, since the mass of the strange quark is of the order of the QCD scale-parameter, it is less obvious that the chiral expansion at next-to-leading order will be sufficiently accurate. Much improved empirical information of these threshold parameters might therefore lead to a better understanding of

*Fig. 1:  $\pi\pi$  scattering phase shift  $\delta_0^0(s)$ . The dashed line gives the tree result and the dashed-dotted the one-loop prediction. Also shown is the Roy equation band. The data can be traced back from ref.[12]. The double-dashed line corresponds to the one-loop result based on another definition of the phase-shift which differs at order  $E^6$  from the one leading to the dashed-dotted line (and thus gives a measure of higher order corrections). On the right side of the hatched area, the one-loop corrections exceed 50 per cent of the tree result.*

the three flavor CHPT. Another possibility is that the threshold of  $\pi K$  scattering at 635 MeV is already so high that one has to connect CHPT constraints with dispersion theory. This concept has been investigated in detail by Dobado and Pelaez [14] and certainly improves the prediction in the P-wave drastically. Another way of extending the EFT through the implicit inclusion of resonance degrees of freedom is discussed in ref.[49]. On the experimental side, a measurement of  $\pi K$  molecule decays would certainly help to clarify the situation [11].

### **3. TWO LOOPS AND BEYOND I: SCALAR FORM FACTOR**

The simplest object to study in detail the strong pionic final state interactions in the isospin zero S-wave is a three-point function, namely the so-called scalar form factor

(ff) of the pion,

$$\langle \pi^a(p') \pi^b(p) | \hat{m}(\bar{u}u + \bar{d}d) | 0 \rangle = \delta^{ab} \Gamma_\pi(s) M_\pi^2 \quad (10)$$

with  $s = (p' + p)^2$ . To one loop order, the scalar ff  $\Gamma_{\pi,2}(s)$  has been given in ref.[3]. As shown in fig.2, closely about the two-pion cut, the real as well as the imaginary part of the one loop representation are at variance with the empirical information obtained from a dispersion-theoretical analysis [15]. However, unitarity allows one to write down a two-loop representation [16],

$$\Gamma_\pi(s) = d_0 + d_1 s + d_2 s^2 + \frac{s^3}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^3} \frac{\sigma(s')}{s' - s} \left\{ T_{0,2}^0 (1 + \text{Re}\Gamma_{\pi,2}) + T_{0,4}^0 \right\} \quad (11)$$

where  $T_{0,2}^0$  and  $T_{0,4}^0$  are the tree and one loop representations of the  $\pi\pi$  S-wave, isospin zero scattering matrix. Notice that the imaginary part of  $\Gamma_\pi(s)$  to two loops is entirely given in terms of known one loop amplitudes. The three subtraction constants appearing in (11) can be fixed from the empirical knowledge of the normalization, the slope and the curvature of the scalar ff at the origin. In the chiral expansion, these numbers are combinations of two low-energy constants from  $\mathcal{L}_{\text{eff}}^{(4)}$  and two from  $\mathcal{L}_{\text{eff}}^{(6)}$ .

The turnover of the scalar ff at around 550 MeV can be understood if one rewrites (11) in an exponential form,

$$\text{Re}\Gamma_\pi(s) = P(s) \exp[\text{Re}\Delta_0(s)] \cos \delta_0^0 + \mathcal{O}(E^6) \quad (12)$$

with  $\text{Im} \Delta_0(s) = \delta_0^0 + \mathcal{O}(E^6)$  fulfilling the final-state theorem at next-to-leading order. Although this representation is not unique, it allows to understand the vanishing of  $\text{Re}\Gamma_\pi(s)$  at 680 MeV since the phase (in the loop approximation) passes through  $90^\circ$  at this energy thus forcing the turnover. Expanding  $\cos \delta_0^0 = 1 - (\delta_0^0)^2 + \dots = 1 + \mathcal{O}(s^2/F_\pi^4)$  it becomes clear why this behaviour can only show up at two loop order (and higher). One can do even better and sum up all leading and next-to-leading logarithms by means of an Omnès representation [16]. This leads to a further improvement in  $\text{Re}\Gamma_\pi(s)$  and allows to understand that the very accurate two loop result for  $\text{Im}\Gamma_\pi(s)$  is not spoiled by higher orders, these can be estimated from the improved chiral expansion of the scalar ff and are found to be small below 550 MeV. The physics behind all this is that the two-loop corrections lead to the two-pion cut with proper strength which dominates the scalar ff below 600 MeV. To go further one would have to include inelasticities (which start at order  $E^8$ ), in particular the strong coupling to the  $\bar{K}K$  channel. It is also worth pointing out that the scalar ff can only be represented by a polynomial below  $s = 4M_\pi^2$ .

*Fig. 2: Scalar form factor of the pion. The curves labelled '1', '2', 'O' and 'B' correspond to the chiral prediction to one-loop, to two-loops, the modified Omnès representation and the result of the dispersive analysis, respectively. The real part is shown in (a) and the imaginary part in (b).*

Notice that in this energy range the normalized scalar ff varies from 1 to 1.4, signaling a large scalar radius of the pion. For comparison, the vector ff changes from 1 to 1.15 for  $0 \leq s \leq 4M_\pi^2$ . In this way, unitarity allows to extend the range of CHPT, however, one has to be able to fix the pertinent subtraction constants (which is the equivalent to determining the corresponding low-energy constants).

#### 4. TWO LOOPS AND BEYOND II: TWO-PHOTON FUSION

Another reaction which has attracted much attention recently is  $\gamma\gamma \rightarrow \pi^0\pi^0$  in the threshold region. It belongs to the rare class of processes which are vanishing at tree level (since the photon can only couple to charged pions, one needs at least one loop) and do not involve any of the low-energy couplings from  $\mathcal{L}_{\text{eff}}^{(4)}$  at one loop order. Some years ago, Bijmens and Cornet [17] and Donoghue, Holstein and Lin [18] calculated the



one-loop cross section and found that it is at variance with the Crystal ball data [19] even close to threshold (see fig.3). Denoting by  $s$  the cms energy squared, the amplitude can be written in terms of a single invariant function (at order  $E^4$ )

$$\begin{aligned}
\mathcal{A}(\gamma\gamma \rightarrow \pi^0\pi^0) &= A(s, t, u) \left[ -\frac{s}{2} \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \right] \\
A(s, t, u) &= A^\pi(s, t, u) + A^K(s, t, u) \\
A^\pi(s, t, u) &= ie^2 \frac{1}{4\pi^2 F_\pi^2} \left[ 1 - \frac{M_\pi^2}{2} \right] \left[ 1 + \frac{M_\pi^2}{s} \ln^2 Q_\pi \right] \\
A^K(s, t, u) &= ie^2 \frac{1}{16\pi^2 F_\pi^2} \left[ 1 + \frac{M_K^2}{s} \ln^2 Q_K \right] \\
Q_i &= \frac{\sqrt{s_i - 4} + \sqrt{s_i}}{\sqrt{s_i - 4} - \sqrt{s_i}}, \quad s_i = \frac{s}{M_i^2} \quad (i = \pi, K)
\end{aligned} \tag{13}$$

with the contribution to the pion loops being completely dominant. The apparent discrepancy between the one-loop prediction and the data (cf. Fig.3) even very close to threshold was long considered a severe problem for CHPT. Notice that for charged pion production, this problem does not occur since there is a dominant  $\mathcal{O}(E^2)$  contribution which already is close to the data. Furthermore, in the threshold region the total cross section for  $\gamma\gamma \rightarrow \pi^+\pi^-$  is approximately two order of magnitude larger than for the double neutral pion production, i.e. one is after a small effect.

In fact,  $\gamma\gamma \rightarrow \pi^0\pi^0$  is another case where one has to account for the strong pionic final state interactions. At 400 MeV, one has

$$\left( \frac{\sigma^{\text{exp}}}{\sigma^{1\text{-loop}}} \right)^{1/2} = 1.3 \tag{14}$$

which is a typical correction in this channel (see discussion above on  $a_0^0$  and the scalar ff). In fact, dispersion theoretical calculations supplemented with current algebra constraints by Pennington [20] tend to give the trend of the data (see the shaded area in fig.3). An improved combination of chiral machinery and dispersion theory has been given by Donoghue and Holstein [21]. Even better, Bellucci, Gasser and Sainio [22] have performed a full two loop calculation. It involves some massive algebra and three new low-energy constants have been estimated from resonance exchange (the main contribution comes from the  $\omega$ ). These couplings play, however, no role below 400 MeV. The solid line in fig.3 shows the two-loop result for the central values of the coupling constants. One finds a good agreement with the data up to  $E_{\pi\pi} = 700$  MeV. This resolves the long-standing discrepancy between the chiral prediction and the data in the

*Fig. 3: Cross section for  $\gamma\gamma \rightarrow \pi^0\pi^0$ . The chiral one and two loop predictions are given by the dotted and the solid line, in order. The hatched area is a dispersion-theoretical fit. The Crystal ball data are also shown. From [22].*

threshold region. For a more detailed discussion of these topics and the related neutral pion polarizabilities, I refer to ref.[22].

Another topic I briefly want to mention in connection with large unitarity corrections is the radiative kaon decay  $K_L \rightarrow \pi^0\gamma\gamma$  which has no tree-level contribution and is given by a finite one-loop calculation at order  $E^4$  [23]. The predicted two-photon invariant mass spectrum turned out to be in amazing agreement with the later measurements [24]. However, the branching ratio which is also predicted was found about a factor three too small. Again, unitarity corrections work in the right direction. In recent work by D'Ambrosio and collaborators [25] and later by Cohen, Ecker and Pich [26] as well as Kambor and Holstein [27] it is shown that unitarity corrections (eventually supplemented by a sizeable  $E^6$  vector meson exchange contribution) can indeed close the gap between the empirical branching ratio and the CHPT prediction though not completely. These calculations are, however, not taking into account all effects beyond  $E^4$  but they underline the importance of making use of dispersion theory in connection with CHPT.

## 5. TWO LOOPS AND BEYOND III: $K_{\ell 4}$ -DECAYS

As already mentioned,  $K_{\ell 4}$ -decays give information concerning the  $\pi\pi$  phase shifts close to threshold (for a good but somewhat old review, see ref.[28]). Because of the  $\Delta I = 1/2$  rule the two pions in the final state can only have total isospin zero or one. In principle, the energy of the two pions  $\sqrt{s_\pi}$  lies between  $2M_\pi$  and  $M_K - m_\ell$ , with  $m_\ell$  the mass of the corresponding lepton (which in the case of the electron can mostly be neglected). Due to phase space, however, only the first 100 MeV above two-pion threshold are really available and one is thus sensitive to the phase difference  $\delta_0^0(s_\pi) - \delta_1^1(s_\pi)$  and  $\delta_1^1(s_\pi)$  stays below  $2^\circ$  for  $\sqrt{s_\pi} < 380$  MeV, i.e. one essentially measures the isospin zero, S-wave. In the more refined treatment discussed below, one takes into account all partial waves allowed. The largest data sample presently available are the 30000  $K_{e4}$  decays measured and analyzed by the CERN-Geneva group [29]. I refer to that reference for a detailed account of the measured distributions and conclusions drawn at that time.

To be specific, consider the decay  $K^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ . The transition matrix-element factors into a leptonic times a hadronic current

$$T = \frac{G_F}{\sqrt{2}} V_{us}^* \bar{u}(p_\ell) \gamma_\mu (1 - \gamma_5) \nu(p_\nu) < \pi^+(p_1) \pi^-(p_2) | I_\mu^{4-i5}(0) | K^+(p) > \quad I = V, A \quad (15)$$

with  $G_F$  the Fermi constant,  $V_{us}$  the pertinent entry in the CKM matrix and the leptonic current is completely known. In contrast, the hadronic ME is parametrized in terms of four form factors, three related to the axial-vector current  $A_\mu$  (denoted  $F, G$  and  $R$ ) and one related to the vector current  $V_\mu$  (denoted  $H$ )

$$\begin{aligned} V_\mu &= -\frac{H}{M_K^3} \epsilon_{\mu\nu\rho\sigma} (p_\ell + p_\nu)^\nu (p_1 + p_2)^\rho + (p_1 - p_2)^\sigma \\ A_\mu &= -\frac{i}{M_K^3} \left[ (p_1 + p_2)_\mu F + (p_1 - p_2)_\mu G + (p_\ell + p_\nu)_\mu R \right] \end{aligned} \quad (16)$$

Clearly, these form factors contain the hadronic physics. The ff  $H$  is obviously related to the chiral anomaly and will be discussed later (since the ME of the vector current is proportional to the totally antisymmetric tensor in four dimensions). The ff  $R$  is only relevant for heavy leptons, say for the muon. I will not discuss it in what follows.

The chiral expansion for the ffs  $F, G$  and  $H$  to leading order  $E^2$  was first given by Weinberg [30] and reads

$$F = G = \frac{M_K}{\sqrt{2}F_\pi} = 3.74 \quad H = 0 \quad (17)$$

i.e. at that order one sees no momentum dependence (cf. the discussion of the scalar ff of the pion in section 3). The one-loop representation for  $F$  takes the form [31,32]

$$F(s_\pi, t, u) = \frac{M_K}{\sqrt{2}F_\pi} \left[ 1 + \frac{1}{F_\pi^2} (U_F + P_F + C_F) \right] + \mathcal{O}(E^6) \quad (18)$$

with  $t = (p - p_1)^2, u = (p - p_2)^2$  and the expansion for  $G$  looks similar. At next-to-leading order, one has unitarity corrections ( $U_F$ ) from the one loop graphs, in this particular case they are proportional to the tree level prediction for  $\delta_0^0(s_\pi)$ . The low-energy constants  $L_i$  are subsumed in the polynomial piece  $P_F$  and  $C_F$  contains chiral logs. As it turns out, these ffs are only sensitive to  $L_{1,2,3}$  and thus the other  $L'_i$ s contributing are taken from previous determinations. The  $E^4$  prediction for  $H$  will be discussed later.

Now we can ask the question whether this one loop result for the ffs  $F$  and  $G$  will be sufficiently accurate to pin down the low-energy constants  $L_{1,2,3}$  and therefore the  $\pi\pi$  phases? For that, we compare with the data of ref.[29] at threshold,

$$F^{\text{thr}} = 3.74[1 + m_q(\alpha + \beta L_1 + \gamma L_2 + \delta L_3) + \dots]^2 5.59 \pm 0.14 \quad (19)$$

where the ellipsis stands for one and higher loop contributions. Clearly, the empirical number is a factor 1.5 larger than the tree prediction. The  $L_i$  can only be determined precisely if one can estimate the higher order corrections. This has been done by Bijmens, Colangelo and Gasser [33] who write down a dispersive representation for  $F$  (and also  $G$ ),  $F = f_s \exp(i\delta_0^0)$  in the spirit of section 3. The two-pion cut is taken out by a modified Omnès function and the remaining polynomial piece is smooth,  $\tilde{f}_s = a + bm_q$  and terms of order  $m_q^2$  have been neglected. The details of this procedure are spelled out in ref.[33]. While the resulting numbers for the  $L_{1,2,3}$  are sensitive to the inclusion of higher orders, one finds a beautiful consistency between the  $\pi\pi$  threshold parameters derived first only from the  $K_{e4}$  data of ref.[29] and second by adding the existing threshold  $\pi\pi$  data from other reactions. This is shown in table 1 together with the empirical numbers from Petersen [34]. It should also be stressed that the one-loop plus unitarization calculation leads to a much improved description of the  $\pi\pi$  D-wave scattering lengths. These were originally used to pin down the values for  $L_1$  and  $L_2$  [3,6] and that procedure was often criticized since the empirical values have large uncertainties. In ref.[33], many other

chiral predictions are given and I refer the reader for all the details to that paper.

	$K_{e4}$ data	$K_{e4} + \pi\pi$ data	Exp.
$a_0^0$	0.20	0.20	$0.26 \pm 0.05$
$-10 a_0^0$	0.41	0.41	$0.28 \pm 0.12$
$10 a_1^1$	0.37	0.37	$0.38 \pm 0.02$
$100 a_2^0$	0.18	0.18	$0.17 \pm 0.03$
$100 a_2^2$	0.21	0.20	$0.13 \pm 0.30$

*Table 1:  $\pi\pi$  scattering lengths in appropriate units of inverse pion masses. The numbers are taken from ref.[33] and represent the calculation including higher loop effects via unitarization.*

## 6. ANOMALIES: GENERAL REMARKS

As already mentioned, there are processes which are proportional to the totally antisymmetric tensor in four dimensions. These are related to anomalies, in our case the so-called chiral anomaly. In this section, I will give a short discussion about the meaning of anomalies in QFTs. For more details, I refer to the monograph by Treiman, Jackiw, Zumino and Witten [35] (and refs. therein).

First, I have to define what an anomaly is. One speaks of an *anomaly* if a *classical Lagrangian symmetry* is *broken upon quantization*. Although anomalies are related to short distance phenomena, they show up most clearly at long wave lengths (as I will show in what follows). Furthermore, in QFTs such effects are quite normal, remember that anomaly cancelation plays a central role in the quantization of field theories like the standard model. To get an idea, let me briefly give a field theoretic view of anomalies in the path integral formalism following the work of Fujikawa [36]. Consider a Lagrangian  $\mathcal{L}(\Psi, \bar{\Psi}, \dots)$  which is invariant under transformations like

$$\Psi \rightarrow \Psi' = \exp[iS] \Psi, \dots \quad (20)$$

with  $S = S_a T_a$  and  $T_a$  the generators of the corresponding algebra, i.e.  $\mathcal{L}(\Psi', \bar{\Psi}', \dots) = \mathcal{L}(\Psi, \bar{\Psi}, \dots)$ . At the quantum level, we consider the generating functional

$$\mathcal{Z} = \int [d\Psi][d\bar{\Psi}][\dots] \exp \left\{ i \int d^4x \mathcal{L}(\Psi, \bar{\Psi}, \dots) \right\}. \quad (21)$$

Clearly, under the symmetry transformation related to  $S$  the measure might change,

$$[d\Psi][d\bar{\Psi}] \rightarrow [d\Psi'][d\bar{\Psi}']|\mathcal{J}| \quad (22)$$

so if the Jacobian is not equal one,  $|\mathcal{J}| \neq 1$ , we encounter an anomaly, i.e. the classical Lagrangian symmetry is broken. As an example, consider massless QED where  $\Psi$  denotes an isodoublet (say of u and d quarks) and  $A_\mu$  the photon field (U(1) gauge field),

$$\mathcal{L}(\Psi, \bar{\Psi}, A_\mu) = \bar{\Psi}(i\not{\partial} - QA_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (23)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the photon field strength tensor and  $Q$  is the (quark) charge matrix. Under axial transformations  $\Psi \rightarrow \exp[i\kappa\gamma_5]\Psi$  the Lagrangian is obviously invariant and the corresponding Noether current  $J^{\mu 5} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$  is conserved,  $\partial_\mu J^{\mu 5} = 0$ . Upon quantization, the measure picks up a nontrivial Jacobian,  $[d\Psi'][d\bar{\Psi}'] = [d\Psi][d\bar{\Psi}]\exp[-2i\text{Tr}(\kappa\gamma_5)]$  which leads to a non-vanishing derivative of the axial current,

$$\partial_\mu J^{\mu 5} = \frac{\alpha}{12\pi} N_c \epsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} \quad (24)$$

with  $\alpha = 1/137$  the fine structure constant and  $N_c$  denotes the number of colors. This is, indeed, the original Adler–Bell–Jackiw [37] anomaly which leads to a finite lifetime for the  $\pi^0$  decay into two photons,

$$\Gamma(\pi^0 \rightarrow 2\gamma) = \frac{\alpha^2 N_c^2 M_\pi^3}{576\pi^3 F_\pi^2} = (7.6 \text{ eV}) \frac{N_c^2}{9} \quad (25)$$

which compared with the empirical value of  $(7.7 \pm 0.6) \text{ eV}$  is one of the strongest arguments that the number of colors is indeed three,  $N_c = 3$ .

## 7. THE CHIRAL ANOMALY A LA WESS–ZUMINO–WITTEN

The chiral anomaly was first discussed by Wess and Zumino [38] in the context of anomalous Ward identities and later given a beautiful geometrical interpretation by Witten [39]. I will essentially only give a pedagogical treatment of the topic following the review [40]. To be specific, let us consider the first term in the energy expansion of eq.(4),

$$\mathcal{L}^{(2)} = \frac{F_\pi^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \quad (26)$$

where  $U(x)$  is an element of SU(3) and subsumes the Goldstones (there is no chiral anomaly for the two flavor case). As is QCD,  $\mathcal{L}^{(2)}$  is invariant under parity,

$PU(\vec{x}, t)P^{-1} = U^\dagger(-\vec{x}, t)$ . However, besides that,  $\mathcal{L}^{(2)}$  has two extra symmetries, i.e. it is invariant under

$$U(\vec{x}, t) \rightarrow U(-\vec{x}, t) \quad \text{and} \quad U(\vec{x}, t) \rightarrow U^\dagger(\vec{x}, t) \quad . \quad (27)$$

It is easily understood that this means that intrinsic parity or the number of Goldstone bosons modulo two is conserved. Intrinsic parity is defined as follows. For a true (pseudo) tensor of rank  $k$ , intrinsic parity  $P_I$  is plus (minus) one. So scalars, polar vectors, ... have  $P_I = +1$  whereas pseudoscalars, axial vectors, ... have  $P_I = -1$ . Furthermore, intrinsic parity is a multiplicative quantum number. Typical processes conserving  $P_I$  (and thus the number of Goldstones modulo two) are  $\pi\pi \rightarrow \pi\pi$ ,  $\gamma\pi \rightarrow \pi$ ,  $\gamma\gamma \rightarrow \pi\pi$  or  $\eta \rightarrow 3\pi$ . Intrinsic parity is violated for  $\pi^0 \rightarrow 2\gamma$ ,  $K^+K^- \rightarrow \pi^+\pi^-\pi^0$ ,  $\gamma \rightarrow \pi^+\pi^-\pi^0$  or the ME  $\langle \pi\pi | V_\mu | K \rangle$  encountered in section 5. Similar observations can be made for the terms of order  $E^4$  (and higher) in eq.(4). So the moral is that while the effective Lagrangian does conserve  $P_I$ , QCD does not. To break the redundant symmetries on the level of the equation of motion for  $L_\mu = U^\dagger \partial_\mu U$  derived from  $\mathcal{L}^{(2)}$ , one can easily write down an extra term [39],

$$\frac{i}{2} F_\pi \partial^\mu L_\mu + \lambda \epsilon^{\mu\nu\alpha\beta} L_\mu L_\nu L_\alpha L_\beta + \dots = 0 \quad (28)$$

with the constant  $\lambda$  to be fixed later. As pointed out by Witten, this can not be written in terms of a four-dimensional Lagrangian but rather as an integral over a 5-dimensional sphere which bounds space-time (I write down only the term for the interactions between the Goldstone bosons),

$$\Gamma_{\text{WZW}} = -\frac{in}{240\pi^2} \int_{S^5} d^5x \epsilon^{\mu\nu\alpha\beta\gamma} \text{Tr} [L_\mu L_\nu L_\alpha L_\beta L_\gamma] \quad . \quad (29)$$

By topology,  $n$  has to be an integer number. It can be fixed when one gauges  $\Gamma_{\text{WZW}}$  (correctly done first in refs.[41]) and compares with the result for  $\pi^0 \rightarrow 2\gamma$ , eq.(24). This leads to the identification

$$n = N_c \quad (30)$$

and therefore the effective meson Lagrangian still knows about the number of colors of QCD, an amazing result. From the point of chiral counting, the Wess-Zumino-Witten term is of order  $E^4$ . At this leading order, it is uniquely fixed, i.e. does not introduce any novel low-energy constant. Let me finish this section by one curious experimental result. From the gauged WZW action, one can immediately derive the amplitude for  $\gamma \rightarrow 3\pi$ ,  $A(\gamma \rightarrow 3\pi) = eN_c/12\pi^2 F_\pi = 9.7 \text{ GeV}^{-3}$ . In ref.[42], an empirical determination of

this quantity was presented. The measurement was based on the production of pions in the virtual field of a nucleus,  $\pi A \rightarrow \pi\pi A$  via one photon exchange. Interpolating to the low energy limit, one arrived at  $12.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3}$  [42] which is at variance with the theoretical prediction. However, a remeasurement as well as more thorough theoretical calculations are called for before one can draw a final conclusion. In fact, if one considers the process  $\gamma N \rightarrow \pi\pi N$  (here,  $N$  denotes the nucleon) [43], there are many other competing diagrams and it is not yet clear how cleanly one could separate out the anomalous  $\gamma 3\pi$  vertex.

## 8. SIGNALS OF THE CHIRAL ANOMALY

In this section, I will briefly talk about a few intrinsic parity violating reactions related to decays of pions, kaons and etas. An older review is ref.[44] and a fresh look in view of CHPT has recently been given by Bijmans [45] (which contains much more details than given here).

A first example is the decay mode  $\eta \rightarrow \pi^+\pi^-\gamma$ . From the WZW action, one predicts  $\Gamma(\eta \rightarrow \pi^+\pi^-\gamma) = 35 \text{ MeV}$  to be compared with the PDG value of  $58 \pm 6 \text{ eV}$ . However, there is large vector meson contribution starting at order  $E^6$  which goes in the right direction. Now let me return to the  $K_{e4}$ -decays discussed previously. As already noted, the contribution of the vector current to the hadronic ME is of anomalous nature and thus only starts to contribute at order  $E^4$  and is entirely given in terms of known parameters [31,32]

$$H = -\frac{\sqrt{2}M_K^3}{8\pi^2 F_\pi^3} + \mathcal{O}(E^6) = -2.66 \quad (31)$$

which compares well with the empirical number [29],

$$H_{\text{exp}} = -2.68 \pm 0.68 \quad (32)$$

if one uses the pion decay constant. It would also be legitimate to use  $F_K = 1.22 F_\pi$  here, thus reducing the theoretical prediction by a factor 1.8. The order  $E^6$  corrections have been calculated and found to be small if one estimates the appearing low-energy constants using vector mesons only [46]. Most amazing, however, is the fact that the empirical result checks indeed the sign of the chiral anomaly. Another wide field to study the chiral anomaly are radiative pion and kaon decays such as  $\pi \rightarrow e\nu_e\gamma$ ,  $K \rightarrow l\nu_l\gamma$  or  $K \rightarrow \pi l\nu_l\gamma$ . For a detailed discussion of these, I refer the reader to the updated version of the DAΦNE handbook [47]. Finally, I wish to mention non-leptonic radiative  $K$ -decays. Examples are  $K_L \rightarrow \pi^+\pi^-\gamma$ ,  $K^+ \rightarrow \pi^+\pi^0\gamma$ ,  $K \rightarrow \pi\pi\gamma\gamma$  or  $K \rightarrow 3\pi\gamma(\gamma)$ . These have



been studied extensively by Ecker, Neufeld and Pich [48] (see also the references given therein). Apart from the reducible amplitudes, which can directly be derived from the WZW functional (and are thus unambiguous), there are also so-called direct contributions which induce some theoretical uncertainties. In certain channels, one furthermore has to account for the  $E^6$  contributions, which come from  $\eta - \eta'$  mixing and vector meson exchange. This is a complementary field of testing the chiral anomaly, which is and will be exploited in more detail in the future. I finish this section with the remark that at present the "standard" anomalous strong process, namely  $K^+ K^- \rightarrow \pi^+ \pi^- \pi^0$  has not yet been observed.

## 9. WHY?

In this lecture I could only give a glimpse of the many facets of chiral perturbation theory. Instead of repeating what was already said, let me briefly remind you of why all these calculations are done. Clearly, the effective chiral Lagrangian approach gives us some insight about some *fundamental parameters* of *QCD*. In the introduction I mentioned already the ratios of the light quark masses and in section 2 I pointed out that refined measurements of e.g. the  $\pi\pi$  threshold parameters would put stringent test on our understanding of the mode of quark condensation, in particular how large the value of the order parameter  $B$  actually is. Furthermore, the chiral anomaly is a direct consequence of the fact that the standard model is a chiral QFT. At present, not too many experimental tests of this important ingredient of modern particle physics exist. CHPT is the effective field theory of the Standard Model at low energies and has to be subjected to as many empirical tests as possible. In general, calculations to order  $E^4$  are already accurate, however, as discussed here, there exist circumstances when one has to work harder. These are essentially related to strong pionic final state interactions and can be treated in a combination of dispersion theory with CHPT constraints. We are looking forward to the operation of DAΦNE and it might also be worthwhile to analyze the many K-decays which are on tape from other experiments but are only considered as backgrounds.

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